Analysis of Thermally Loaded Laminated Circular Plates

Report No. 2

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JHRAC Project 70-2 JHR 71-43

August 1971

Abstract

This report presents a summary of the significant research carried out on both layered pavement and plate systems. Most of the layered pavement research has adopted an elasticity or viscoelasticity approach as opposed to employing laminated plate theory. This is the path which realistically must be taken when one is primarily interested in investigating stresses and displacements caused by surface loads. When temperature induced stresses are of main concern, however, laminated plate theory offers many advantages.

In addition to the historical review of the general field, results for the one-and two-layered plate systems are presented. The one-layered case was investigated as a separate entity and also treated as a special case of the two-layered system.

A study of the results shows the effect of the different layer properties on stress conditions and offers a comparison of the one-and two-layered systems. The following trends are apparent:

1. In the one-layered system, $(\sigma_r)_{max}$ occurs at the bottom of the pavement and increases with thickness in almost linear fashion.
2. In the two-layered system, $(\sigma_r)_{max}$ usually increases as the coefficients of thermal expansion of the layers diverge from one another.
3. In the two-layered system, the relative values of the layer thicknesses have only a minor effect on the values of the critical stress magnitudes when $\alpha_1 = \alpha_2$.
4. It is unwise to approximate a two-layered pavement system by a one-layered one when concerned with the evaluation of thermal stresses.

This work is continuing in the Civil Engineering Department of the University of Connecticut under the sponsorship of the Joint Highway Research Advisory Council and the Connecticut Department of Transportation and the University of Connecticut.

Summary of Related Work

Much significant theoretical research concerning elastic and visco-elastic plates and half-spaces has been performed. The solution to the problem of a point loading acting on a homogeneous elastic half-space was presented by Boussinesq[1] in 1885 and the generalized solution for the case of distributed surface loads was solved by Terazawa[2] in 1916. In 1926, Westergaard[3] solved the problem of an elastic plate resting on an elastic "Winkler" foundation. This was probably the first attempt made to predict pavement stresses. The case of an elastic plate resting on an elastic semi-infinite subgrade was handled in 1938 by Hogg[4] and extended later by several other investigators.[5,6]

In 1945, Burmister[7] presented the solution for a two-and three-layer elastic system in which the bottom layer was taken as a semi-infinite half space. This work represented a tremendous improvement over the earlier plate representation of the roadway. Burmister's contribution was subsequently extended by several different investigators. Fox,[8] in 1948, used Burmister's results to evaluate stresses at the interface of a two-layered system subjected to a uniform circular load. Further extensions were made by Hank and Scrivner,[9] Acum and Fox,[10] Schiffman,[11,12] Jones,[13] and Peattie.[14]

Burmister continued to contribute to pavement research during this period. In 1956, he[15] investigated the problem of a compressible soil layer of constant thickness supported by a rigid layer, thus modeling the case of a uniform soil layer resting on solid rock. The following year he published[16] a study of the pavement systems of the WASHO Road Test using his layered theory. In this work an analysis of the effectiveness and the mechanics by which a layered system spreads an applied load was presented.

A more general solution to the elastic layered problem was furnished by Mehta and Veletsos.[17] Their solution applies to an "n" layered system and considers both normal and tangential applied loadings. In 1962, Burmister[18] filled 26 volumes with computer output on a three-layered system covering a wide range of parameters. There has been a small amount of four-layered pavement research conducted. Verstraeten[19] and Barksdale and Leonards[20] presented papers on this subject at the Second International Conference on the Structural Design of Asphalt Pavements in 1967.

Coupled with this development of analytical solutions to elastic layered systems was a correspondingly rapid increase in experimental investigations into the individual and combined behavior of roadway system components. It soon became apparent that, in many situations, pavement systems do not behave elastically but viscoelastically.

The theory of viscoelasticity is still developing. Alfrey[21] contributed greatly to the theory when, in 1944, he formulated his "correspondence principle" for incompressible viscoelastic bodies. This principle was later generalized and extended by Tsien[22] and Lee[23]. In his 1955 paper, Lee presented the solution for a fixed and moving point load on a viscoelastic half-space which was assumed to behave as a Voigt model in shear and to behave elastically in hydrostatic tension or compression. The solution for a viscoelastic plate resting on a viscoelastic foundation was published in 1961 by Pister.[24] The plate was loaded with a uniform circular load and the system was assumed to behave as an incompressible Maxwell material. The same problem was presented in 1965 by Kraft[25] except that the layers were composed of three element models with the volumetric behavior taken as elastic. In all of these early investigations the Laplace transform method of solution was employed.

Using simple discrete spring and dashpot models to characterize viscoelastic material behavior necessarily results in accurate predictions of material behavior only over very short time periods. Measured creep and/or relaxation functions in the form of hereditary integrals represents an alternate approach to the problem. For example, in 1962, Rogers and Lee[26] used the hereditary form of the stress-strain equations to treat the finite deflections of a viscoelastic cantilever beam. The creep-bending of a beam column was investigated by Baltrukonis and Vaishnar[27] in 1965. In 1967, Moavenzadeh and Ashton[28] published an excellent report on the analysis of stresses and displacements in a three-layered viscoelastic system using this hereditary constitutive equation approach. This first investigation was limited to static loading situations. In 1969, however, Elliot and Moavenzadeh[29] investigated the more complicated and more realistic problem of a moving load on a viscoelastic layered system.

In almost all of the references sighted, the thermal stress problem has been ignored and attention focused on load induced stresses and displacements. Some work has been conducted, however, on thermal stresses in single-layered plates. In 1935, Maulbetsch[30] employed classical plate theory to solve for the thermal stresses in simply supported thin plates of various shapes. He considered a linearly varying temperature distribution through the thickness. Goldberg[31] and Forray and Newman[32,33,34] studied circular plates with various edge conditions subjected to a linearly varying temperature distribution. The rectangular plate thermal stress problem was investigated by Williams,[35] Zaid and Forray,[36] Lieb,[37] Rama and Johns,[38] and Chao and Anliker.[39] Various temperature fields and boundaries were considered.

In order to apply plate theory to determine thermal stresses in actual pavements, it becomes necessary to consider a layered plate system. Several investigators have considered the problem of thermal deformations and stresses in layered plates. E.I. Grigulyuk,[40] using small deflection theory, obtained a general formulation for the axisymmetrical bending of bimetallic plates and shells. Another treatment of the same problem, restricted to the case when Poisson's ratio was the same in each layer, was carried out by J.R. Vinson.[41] Numerical results concerning this bimetallic plate problem were presented by V.I. Korolev.[42] K.S. Pister and S.B. Dong[43] derived a general large deflection theory of layered plates. However, no examples were given and the equations are difficult to work with. The governing equations for the bending of a two-layered circular plate subjected to an asymmetric temperature distribution were obtained by J.S. Kao and R.L. Groff,[44] based on small deflection thin plate theory. To the best of the Principal Investigator's knowledge, however, an in depth investigation into the behavior of a three-layered system subjected to thermal loading has not been undertaken.

The Single-Layered System

For completeness, the single-layered case will be solved in two different ways. It will be treated both as a separate problem and as a special case of the two-layered system.

A. The Single-Layered Plate Solution Derived from Basic Principles.

Consider a plate of uniform thickness (Fig. 1) in which the temperature, T, varies only with depth. The total strain is the sum of the thermal strain and the stress induced strain. The stress induced strain may be written as

$$\varepsilon_r' = \varepsilon_r - \alpha T \tag{1}$$

$$\varepsilon_\theta' = \varepsilon_\theta - \alpha T \tag{2}$$

$$\gamma_{r\theta}' = \gamma_{r\theta} \tag{3}$$

in which α is the coefficient of thermal expansion of the plate and T is the difference between the actual temperature and the initial, unstressed temperature.

Assume for a moment that the longitudinal expansion of the plate is entirely suppressed by applying longitudinal compressive stresses. Equations (1) and (2) yield

$$\varepsilon_r' = \varepsilon_\theta' = -\alpha T \tag{4}$$

Hooke's Law is then given as

$$\sigma_r = \frac{E}{1-\nu^2}\,(\varepsilon_r' + \nu\varepsilon_\theta') = -\frac{\alpha T E}{1-\nu} \tag{5}$$

Equation (5) represents the magnitude of the tractions which must be applied to the plate to eliminate longitudinal thermal expansions. The thermal stress in the plate free from external longitudinal forces is is obtained by superimposing on the stresses given in Eq. (5) the stresses due to the application of equal and opposite distributions of forces on the edges. These forces have the resultant

$$\frac{1}{1-\nu} \int_0^h \alpha TE dz$$

in which h is the thickness of the plate. At a sufficient distance from the ends of the plate, the applied tractions will produce uniformly distributed stresses of magnitude

$$\frac{1}{h(1-\nu)} \int_0^h \alpha TE dz$$

Therefore, the thermal stress in a single-layered plate with the boundary conditions as shown in Fig. 1 is

$$\sigma_r = -\frac{\alpha TE}{1-\nu} + \frac{1}{h(1-\nu)} \int_0^h \alpha TE dz \qquad (6)$$

B. The Single-Layered Plate Solution Obtained as a Special Case of the Two-Layered Plate.

The formulation and solution for the two layered plate was presented in Report No. 1, submitted in April of 1971. Upon working with Eqs. (18), (28), (31), (49), (50), (65), and (66) of that report, and setting

$$E_1 = E_2 \text{ and } \alpha_1 = \alpha_2 \qquad (7)$$

it is easily shown that the general expression for σ_r given by Eq. (18) degenerates to Eq. (6). The results obt ined using the general two-layered plate computer program in the light of Eq. (7) agree exactly with the results given by Eq. (6).

Presentation of Results

The results of the single layer investigation will now be discussed and a comparison made of the one- and two-layered systems. For completeness, a summary of the input data used as presented in Report No. 1 will be repeated here. Two different temperature distributions will be considered. In the first case

$$T = 2.5z + 10 \tag{8}$$

simulating an early morning in winter condition. In the second case

$$T = -3.5z + 53.3 \tag{9}$$

which might occur during a sunny afternoon in winter.

Young's modulus is taken as a parabolic function of temperature in the form

$$E_i = g_i T \quad + h_i T + k_i \tag{10}$$

For each pavement considered, both temperature distributions are applied to the system. The maximum radial tensile stress, σ_r, caused by the temperature distributions is plotted in Figs. (2-4).

Fig. (2) plots $(\sigma_r)_{max}$ vs thickness for the single-layered case. The values of the input parameters are listed on the figure. For the temperature distribution given by Eq. (8), the maximum radial bending stress occurred at the top of the pavement, as would be expected. When Eq. (9) was used, $(\sigma_r)_{max}$ occurred at the bottom of the pavement system. However, over the entire range of thicknesses considered (2 inches - 18 inches), the magnitudes of the maximum tensile stresses were largest when the Eq. (9) temperature distribution was employed. This is not always the case in multi-layered systems.

The maximum tensile stress increases with thickness in almost linear fashion. Fig. (2) points out a basic difference between the behavior of a plate under thermal and lateral loadings since, if the plate were loaded only with a lateral load (e.g. a wheel load), the stress would decrease with increasing thickness.

In Fig. (3), the maximum radial tensile stress in a two-layered pavement is plotted against the coefficient of thermal expansion ratio, α_2/α_1 with $E_1 = E_2$. When α_2/α_1 is equal to unity, the properties of the two layers are identical, and $(\sigma_r)_{max}$ is independent of the values of the individual layer thicknesses. On either side of $\alpha_2/\alpha_1 = 1$, the maximum radial tensile stress is very sensitive to the values of the layer thicknesses. For example, when $\alpha_2/\alpha_1 = 0.2$, σ_r varies from 230 psi to 838 psi depending on the values of t_1 and t_2. Furthermore, the general statement can be made that, for a given t_1 and t_2, $(\sigma_r)_{max}$ usually increases as the coefficients of thermal expansion of the layers diverge from one another. For $\alpha_2/\alpha_1 \leq 1$, Eq. (9) is the critical temperature distribution and the maximum radial tensile stress occurs at the bottom of the second layer. For $\alpha_2/\alpha_1 > 1$, Eq. (8) controls and $(\sigma_r)_{max}$ occurs at the top of the first layer.

The relative values of Young's modulus in the layers greatly affect the stress field as evidenced by Fig. (4). In this figure, $(\sigma_r)_{max}$ is plotted against k_2/k_1, in which k_i is the constant term in the parabolic expression for Young's modulus given by Eq. (10). It was found that the relative values of k_i in Eq. (10) are more critical than the parabolic (g_i) or linear (h_i) coefficients in affecting the stress condition of the pavement. In Fig. (4), it is shown that the critical stress increases with c_2/c_1, as would be expected. Note that, unlike Fig. (3), the relative values of t_1 and t_2 have only a minor effect on the values of the critical stress magnitudes. When $k_2/k_1 \leq 0.6$, the maximum radial tensile stress occurs at the top of the first layer except in the $(t_1,t_2) = (5,1)$ case when $(\sigma_r)_{max}$ occurs at the bottom of the first layer. When $k_2/k_1 > 0.6$, $(\sigma_r)_{max}$ always occurs at the bottom of the second layer and is associated with the temperature distribution given by Eq. (9).

A comparison of Figs. (3) and (4) with Fig. (2) illustrates the difference in behavior between a one- and two-layered pavement system. In Figs. (3) and (4), the total thickness of the pavement is taken as six inches. The stress at $\alpha_2/\alpha_1 = 1$ in Fig. (3) and $k_2/k_1 = 1$ in Fig. (4) of course equals the value given in Fig. (2) when $t = 6$. For other values of α_2/α_1 and k_2/k_1, however, the thermal stresses change drastically. It is concluded, therefore, that it is in general unwise to approximate a two-layered pavement by a one-layered system when concerned with thermal stresses. A very slight change in properties (e.g. coefficient of thermal expansion) can result is significant stress increases in the pavement system

Work in Progress

Work on the three-layered pavement system is continuing, and a comparison of the three-layered case with the two-layered pavement analysis will be made and presented in Report #3.

References

1. Boussinesq, J., Application des Potentials, Paris, 1885.

2. Terazawa, Kwan-inchi, Journal of the College of Sciences, Imperial University, Tokyo, December 7, 1916.

3. Westergaard, H.M., "Stresses in Concrete Pavements Computed by Theoretical Analysis", Public Roads, Vol. 7, No. 2, April, 1926.

4. Hogg, A.H.A., "Equilibrium of a Thin Plate, Symmetrically Loaded, Resting on an Elastic Foundation of Infinite Depth", Philosophical Magazine, Vol. 25, p. 576, 1938.

5. Odemark, N., "Investigations as to the Elastic Properties of Soils, and Design of Pavements According to the Theory of Elasticity", Statens Vaginstitut, Moddelande, No. 77, Sweden, 1949.

6. Pickett, G. and Anidaniel, K.V., "Stress in Subgrade Under Rigid Pavement", HRB Proceedings, 1954.

7. Burmister, D.M., "The General Theory of Stresses and Displacements in Layered Soil System, I, II, III", Journal of Applied Physics, Vol. 16, No. 2, pp. 89-96, No. 3, pp. 126-127, No. 5, pp. 296-302, 1945.

8. Fox, L., "Computation of Traffic Stresses in a Simple Road Structure", Road Research Laboratory (England), Road Research Paper 9, London, 1948.

9. Hank, R.J. and Scrivner, F.H., "Some Numerical Solutions of Stresses in Two- and Three-Layered Systems", HRB Proceedings, 28, pp. 457-468, 1948.

10. Acum, W.E.A. and Fox, L., "Computation of Load Stresses in a Three-Layer Elastic System", Geotechnique, Vol. 2, No. 4, pp. 293-300, 1951.

11. Schiffman, R.L., "General Analysis of Stresses and Displacements in Layered Elastic Systems", Proceedings First International Conference on Structural Design of Asphalt Pavements, pp. 365-375, 1962.

12. Schiffman, R.L., "The Numerical Solution for Stresses and Displacements in a Three-Layer Soil System", Proceedings Fourth International Conference on Soil Mechanics and Foundation Engineering, 2, pp. 169-173, 1957.

13. Jones, A., "Tables of Stresses in Three-Layer Elastic Systems", HRB Bulletin 342, 1962.

14. Peattie, K.R., "A Fundamental Approach to the Design of Flexible Pavements", International Conference on the Structural Design of Asphalt Pavements, Proceedings, 1962.

15. Burmister, D.M., "Stress and Displacement Characteristics of a Two-Layer Rigid Base Soil System: Influence Diagrams and Practical Applications", Proceedings, Highway Research Board, Vol. 35, pp. 773-814, 1956.

16. Burmister, D.M., "Evaluation of Pavement Systems of the WASHO Road Test by Layered System Methods", Flexible Pavement Studies and Georgia Design Practice, Highway Research Board, Bulletin 177, Washington, D.C., pp. 26-54, 1957.

17. Mehta, M.R. and Veletsos, A.S., "Stresses and Displacements in Layered Systems", Structural Resaarch Series No. 178, University of Illinois, Urbana, Ill., June, 1959.

18. Burmister, D.M., "Flexible Pavement Design for Airfields", Interim Report on Contract NBy-13009 to Department of the Navy, Bureau of Yards and Docks, Columbia University, New York, N.Y., May 12, 1962. (Reference: B-410-fk:mvs, NBy-13009.)

19. Verstraeten, J., "Stresses and Displacements in Elastic Layered Systems", Proceedings, Second International Conference on the Structural Design of Asphalt Pavements, University of Michigan, Ann Arbor, Mich., pp. 223-238, Aug. 7-11, 1967.

20. Barksdale, R.D. and Leonards, G.A., "Predicting Performance of Bituminous Surfaced Pavements", Proceedings, Second International Conference on the Structural Design of Asphalt Pavements, University of Michigan, Ann Arbor, Mich., pp. 251-276, Aug. 7-11, 1967.

21. Alfrey, T., "Nonhomogeneous Stress in Viscoelastic Media", Quarterly Applied Mathematics, 2, pp. 113-119, 1944.

22. Tsien, H.S., "A Generalization of Alfrey's Theorem for Viscoelastic Media", Quarterly of Applied Mathematics, Vol. VIII, pp. 104-106, 1950.

23. Lee, E.H., "Stress Analyses in Viscoelastic Bodies", Quarterly Applied Mathematics, Vol. 13, No. 2. p. 183, 1955.

24. Pister, K.S., "Viscoelastic Plates on Viscoelastic Foundations", Journal Engineering Mech. Div., ASCE, pp. 43-54, February, 1961.

25. Kraft, David C., "Analysis of a Two-Layer Viscoelastic System", Proceedings of the American Society of Civil Engineers, Journal of the Engineering Mechanics Division, Vol. 91, No. EM6 Part 1, December, 1965.

26. Rogers, T.G. and Lee, E.H., "On the Finite Deflection of a Viscoelastic Cantilever", Proceedings of the Fourth U.S. National Congress of Applied Mechanics, Vol. 2, pp. 977-987, 1962.

27. Baltrukonis, J.H. and Vaishnar, R.N., "Creep Bending of a Beam-Column of a Viscoelastic Material Governed by an Integral Operator Type of Constitutive Law", Proceedings of the Fourth International Congress on Rheology, Interscience Publishers, New York, p. 195, 1965.

28. Moavenzadeh, F. and Ashton, J.E., "Analysis of Stresses and Displacements in a Three-Layer Viscoelastic System", Research Report, R67-31, Department of Civil Engineering, Materials Research Laboratory, Massachusetts Institute of Technology, Cambridge, Massachusetts, August, 1967.

29. Elliot, J.F. and Moavenzadeh, F., "Moving Load on Viscoelastic Layered Systems, Phase II", Research Report, R69-64, Department of Civil Engineering, Materials Research Laboratory, Massachusetts Institute of Technology, Cambridge, Massachusetts, September, 1969.

30. Maulbetsch, J.L., "Thermal Stresses in Plates", Journal of Applied Mechanics, Vol. 2, No. 4, pp. 141-146, December, 1935.

31. Goldberg, J.E., "Axisymmetric Flexural Temperature Stresses in Circular Plates", Journal of Applied Mechanics, Vol. 20, No. 2, pp. 257-260, June, 1953.

32. Forray, M. and Newman, M., "Axisymmetric Bending Stresses in Solid Circular Plates with Thermal Gradients", Journal of the Aerospace Sciences, Vol. 27, pp. 717-718, September, 1960.

33. Newman, M. and Forray, M., "Bending Stresses Due to Temperature in Hollow Circular Plates, Parts I-III", Journal of the Aerospace Sciences, Vol. 27, pp. 792-793, 870-871, 951-952, 1960.

34. Forray, M. and Newman, M., "Bending of Circular Plates Due to Axisymmetric Temperature Distribution", Journal of the Aerospace Sciences, Vol. 28, No. 10, pp. 773-778, October, 1961.

35. Williams, M.L., "Large Deflection Analysis For a Plate Strip Subjected to Normal Pressure and Heating", Journal of Applied Mechanics, Vol. 22, pp. 458-464, December, 1955.

36. Zaid, M. and Forray, M., "Bending of Elastically Supported Rectangular Plates", Proceedings of the American Society of Civil Engineers, Journal of the Engineering Mechanics Division, Vol. 84, No. EM3, July, 1958.

37. Lieb, B.A.,"Cylindrical Bending of a Heated Long Rectangular Plate", Journal of the Royal Aero. Society, Vol. 65, pp. 26-30, January, 1961.

38. Rama Rao, K. and Johns, D.J., "Some Thermal Stress Analyses for Rectangular Plates", Journal of the Aero. Society of India, Vol. 13, No. 4, pp. 99-104, November, 1961.

39. Chao, C.C. and Anliker, M., "The Elastically Clamped Rectangular Plate with Arbitrary Lateral and Thermal Loading", Journal of Applied Mechanics, Vol. 29, No. 3, pp. 578-580, September, 1962.

40. Grigolyuk, E.I., "Theory of Bimetallic Plates and Shells", Akademiia Nauk, USSR Inst. Mekhaniki, Inzhenernyi Sbornik, Vol. 17, pp. 69-120, 1953.

41. Vinson, Jack R., "Thermal Stresses in Laminated Circular Plates", 3rd U.S. National Congress of Applied Mechanics, June, 1958.

42. Korolev, V.I., "Thin Double-Layer Plates and Shells", Akademiia Nauk, USSR Inst. Mekhaniki, Inzhenernyi Sbornik, Vol. 22, pp. 98-110, 1955.

43. Pister, K.S. and Dong, S.B., "Elastic Bending of Layered Plates", ASCE Journal of the Engineering Mechanics Division, Vol. 85, No. EM4, Oct. 1959.

44. Kao, J.S. and Groff, R.L., "Analysis of Laminated Circular Plates", Presented at ASCE Annual Meeting, April 14-18, 1969, Louisville, Ky.

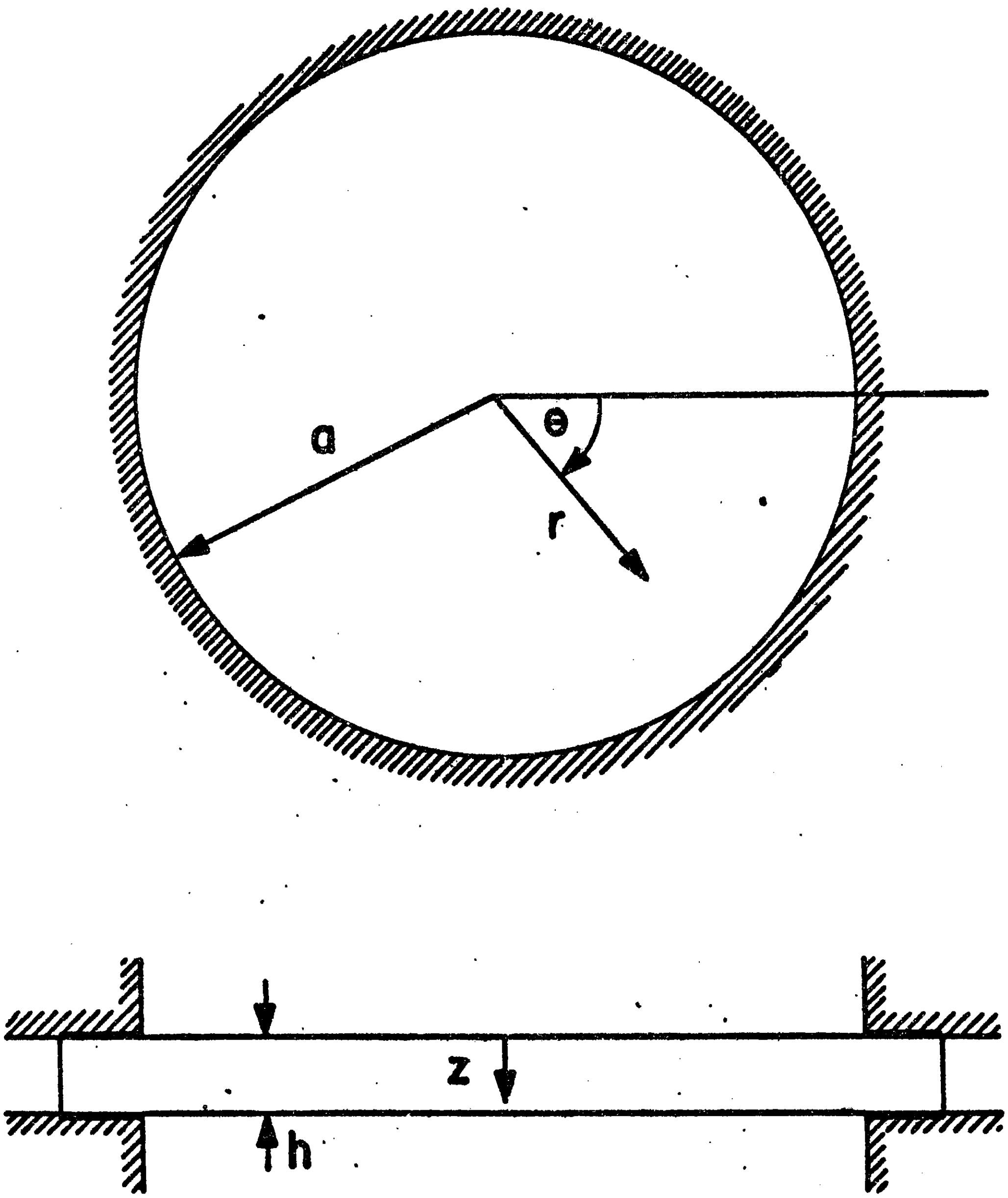

FIGURE 1

Circular Plate

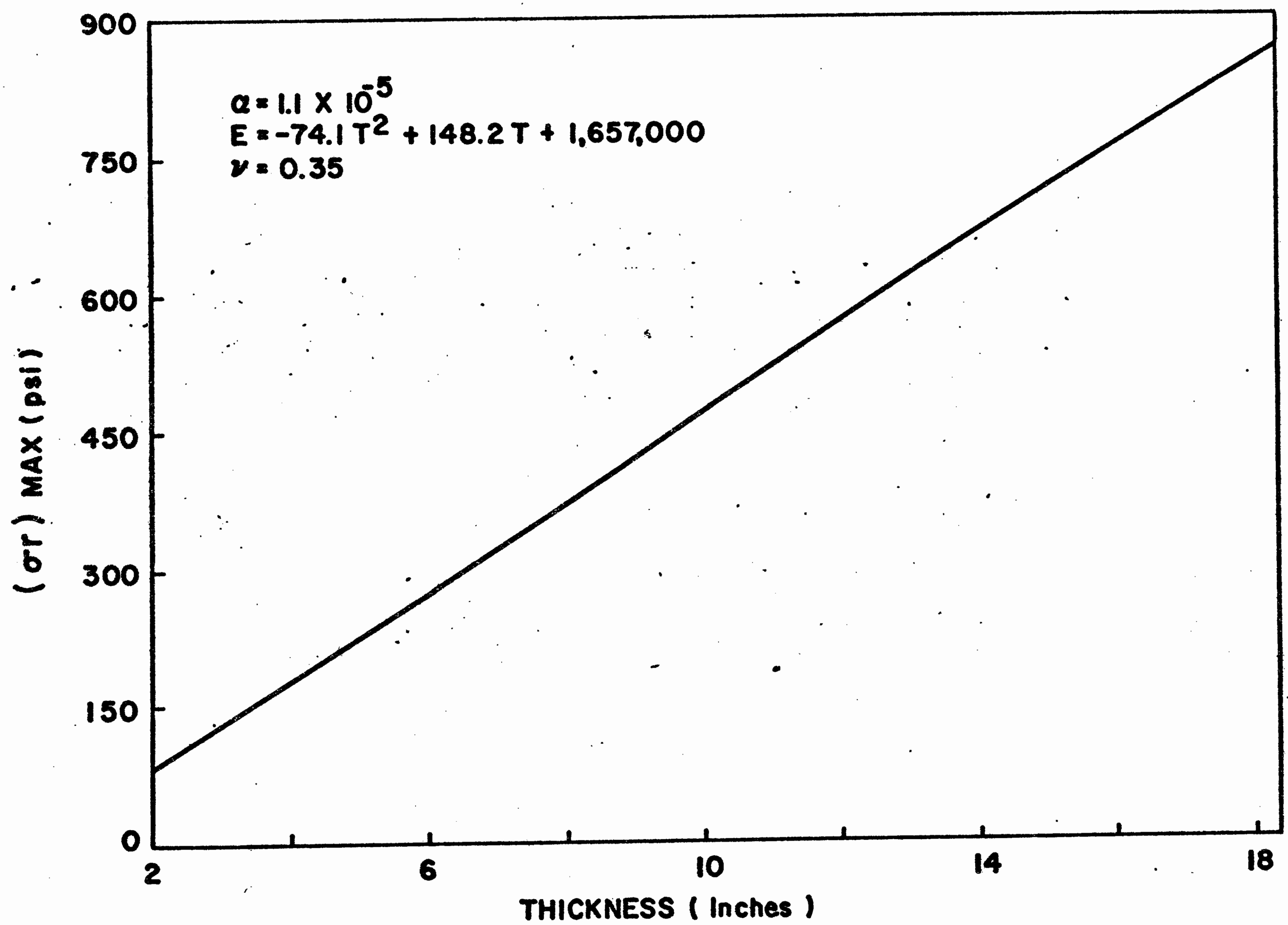

FIGURE 2

$(\sigma_r)_{max}$ versus Thickness

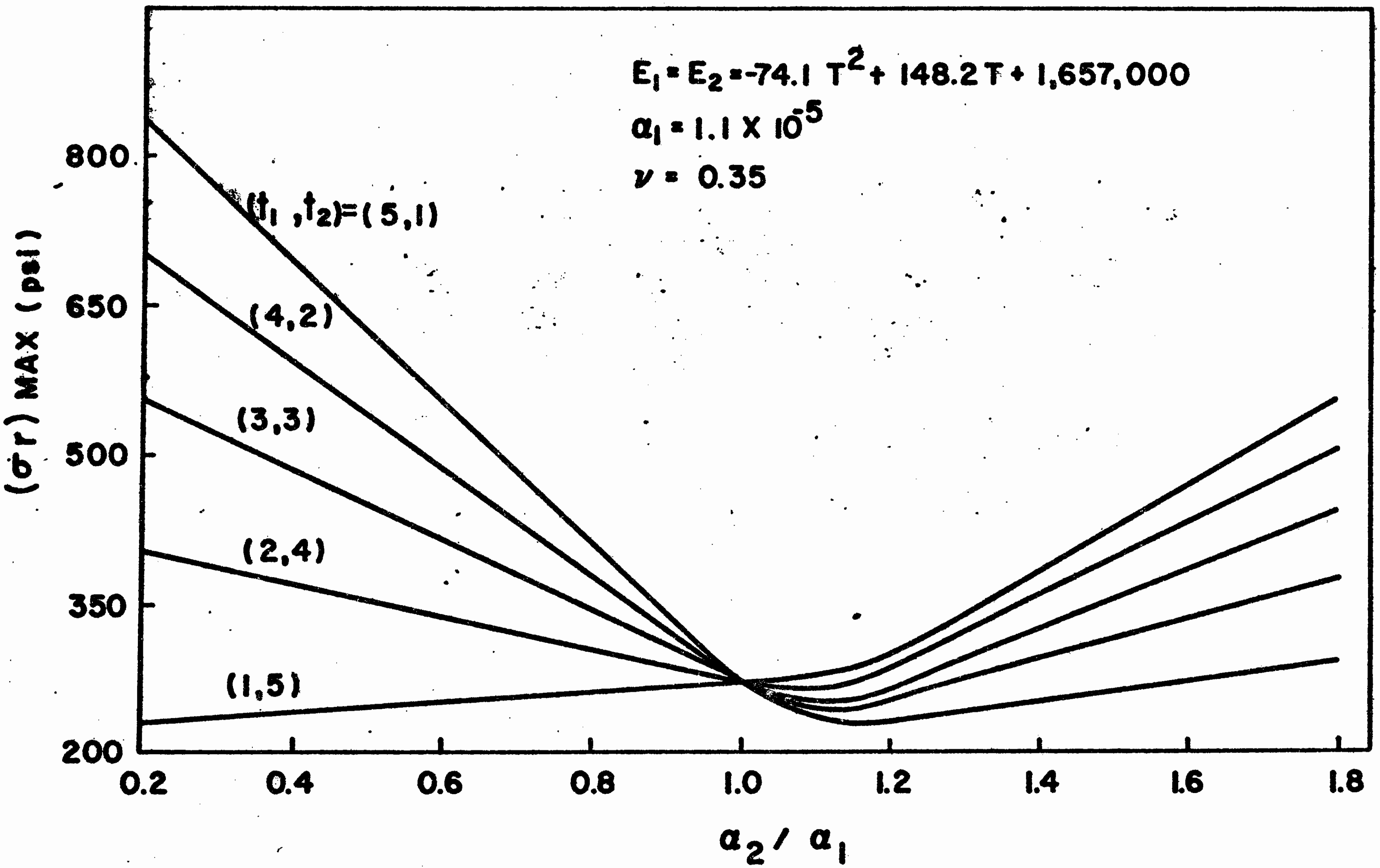

FIGURE 3

$(\sigma_r)_{max}$ versus Coefficient of Thermal Expansion Ratio

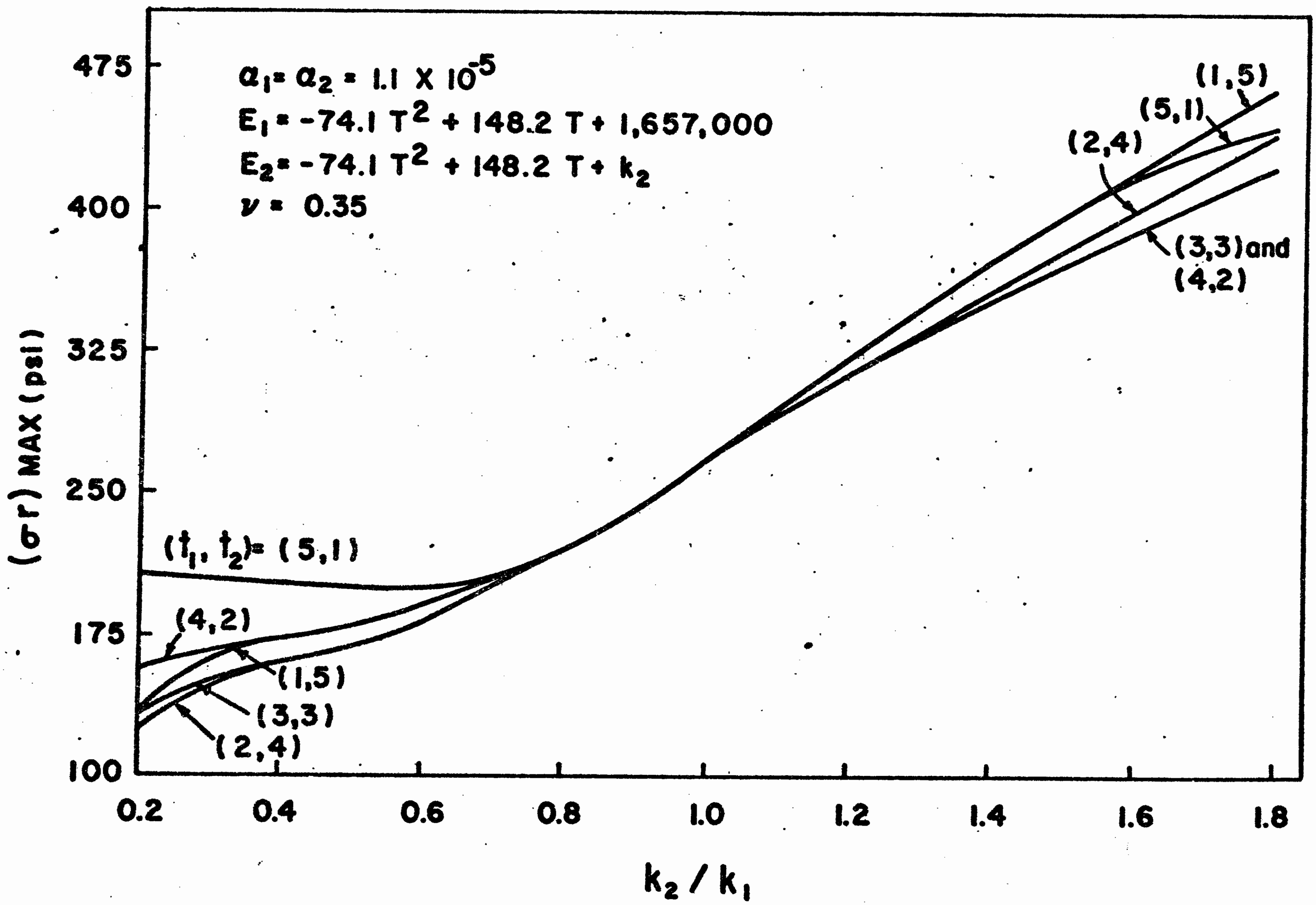

FIGURE 4

$(\sigma_r)_{max}$ versus Young's Modulus Coefficient Ratio